

TOPOLOGICAL GENERATORS OF ABELIAN LIE GROUPS AND HYPERCYCLIC FINITELY GENERATED ABELIAN SEMIGROUPS OF MATRICES

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ABSTRACT. In this paper we bring together results about the density of subsemigroups of abelian Lie groups, the minimal number of topological generators of abelian Lie groups and a result about actions of algebraic groups. We find the minimal number of generators of a finitely generated abelian semigroup or group of matrices with a dense or a somewhere dense orbit by computing the minimal number of generators of a dense subsemigroup (or subgroup) of the connected component of the identity of its Zariski closure.

1. INTRODUCTION

The concept of a hypercyclic abelian finitely generated semigroup of bounded linear operators was introduced by N.S. Feldman in [11]. It generalizes the concept of a hypercyclic operator in linear dynamics. A bounded linear operator on a real or complex topological vector space is called *hypercyclic* if the semigroup generated by this operator has a dense orbit. Hypercyclicity of a single bounded linear operator is a phenomenon that occurs only in infinite dimensional separable spaces, see e.g. [6] or [12], but hypercyclicity of semigroups may occur also on finite dimensional vector spaces, see [11], [14]. A problem raised by Feldman in [11] is to find the minimal number of generators for an abelian semigroup of matrices which is hypercyclic. More specific questions which arise naturally is to solve this problem for special types of matrices. In the same paper Feldman showed that this minimal number of generators for a semigroup of diagonal $n \times n$ matrices with complex entries is $n + 1$. Recently there has been done much research about this subject. We mention only [15], [3], [8], [9], [4], [5], [10] for the

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abelian case and [2], [13] for the non-abelian case. Our methods differ from those in the previously mentioned papers since we use methods from the theories of algebraic groups and algebraic actions.

In this paper we determine the minimal number of generators of a *finitely generated group or semigroup of commuting matrices with real or complex entries* with a dense or a somewhere dense orbit for several classes of matrices. We would like to point out that for this number *there is no difference between dense and somewhere dense orbits and also no difference between the group and the semigroup cases*. This follows from the following theorem which is the basic result of the present paper.

Let V be a finite dimensional real vector space and let S be a subsemigroup of $GL(V)$. For a point $x \in V$ we say that x has a *somewhere dense orbit* if the closure $\overline{S(x)}$ of its orbit $S(x)$ contains a non-empty open subset of V .

Theorem 1.1. *Let S be a finitely generated commutative subsemigroup of $GL(V)$ and let $x \in V$ be a point which has a somewhere dense orbit. Let G be the Zariski closure of S and let G^0 be its connected component of the identity with respect to the Euclidean topology. Then the orbit $G(x)$ of x is an open subset of V , the natural map $G \rightarrow G(x)$, $g \mapsto gx$, is a diffeomorphism and the closure of S is a subgroup of G and contains G^0 .*

We will obtain more precise information about the closure of the orbit Sx in [1]. In particular we will prove that Gx is dense in V for the complex case, which implies the following.

Corollary 1.2. *If additionally V has a structure as a complex vector space such that every $s \in S$ is complex linear, then Sx is dense in V .*

In this paper we show that the orbit Sx is dense in V , if the number of generators of S is minimal among all subsemigroups S of $GL(V)$ having a somewhere dense orbit, except if $\dim V$ is odd. For a precise formulation and further information about this see corollary 5.4.

After Theorem 1.1 it remains to compute the minimal number of generators of a dense subsemigroup (or subgroup) of the abelian Lie group G^0 . This number is one plus the dimension of the following group, namely G^0 modulo its maximal compact subgroup; see section 4. The results of tables 1 and 2 and similar ones follow.

We would like to stress that for complex matrices an orbit of a point is dense if it is somewhere dense but this is not the case for real matrices, see [11]. In the same work Feldman gave a criterion for when a somewhere dense orbit is dense in terms of spectral properties of the

TABLE 1. Minimum number of generators of an abelian (semi)group of real matrices which has a dense or a somewhere dense orbit.

Minimum number of generators of an abelian (semi)group of real matrices		
Commuting	Diagonal	Triangular non-diagonalizable
$(n+2)/2$, if n is even $(n+3)/2$, if n is odd	$n+1$	$n+1$
Triangular Toeplitz non-diagonalizable		
$n+1$		

TABLE 2. Minimum number of generators of an abelian (semi)group of complex matrices which has a dense or a somewhere dense orbit.

Minimum number of generators of an abelian (semi)group of complex matrices		
Commuting	Diagonal	Triangular non-diagonalizable
$n+1$	$n+1$	$n+2$
Triangular Toeplitz non-diagonalizable		
$n+2$		

elements of the semigroup [11, Theorem 5.5]. In section 5 we give an explanation for this difference between the real and the complex case based on a structure theorem for algebraic groups and in a forthcoming paper [1] will give more details.

The contents of the paper are as follows. In section 2 we show that a somewhere dense subsemigroup of a connected abelian Lie group is actually dense. In section 3 we give the proof of theorem 1.1. This proof follows from a basic theorem about actions of algebraic groups [7, § 3.18]. In section 4 we compute the minimal number of generators of a dense subgroup and a dense subsemigroup of a connected abelian Lie group. This is an easy application of Kronecker's theorem. In section 5 we give applications for the case of hypercyclic finitely generated abelian (semi)groups of matrices with various types of generators.

So in this paper we bring together results about the density of subsemigroups of abelian Lie groups (section 2), the minimal number of topological generators of abelian Lie groups (section 4) and a result about actions of algebraic groups (section 3), to solve the problem of determining the minimal number of commuting matrices such that the semigroup they generate has a (somewhere) dense orbit. The results of sections 2 and 4 may be of independent interest.

2. A SOMEWHERE DENSE FINITELY GENERATED SUBSEMIGROUP OF AN ABELIAN CONNECTED LIE GROUP IS DENSE

Theorem 2.1. *Let G be an abelian Lie group whose connected component G^0 is of finite index in G . Let S be a finitely generated subsemigroup of G , which is somewhere dense in G , i.e. its closure \overline{S} contains a non-empty open subset of G . Then \overline{S} is a subgroup of G and contains G^0 .*

We first deal with some special cases.

Lemma 2.2. *Let S be a subsemigroup of the additive group \mathbb{R} , which contains both a negative and a positive real number. Then S is either a discrete cyclic subgroup of \mathbb{R} or S is dense in \mathbb{R} . In particular, the closure of S is a group.*

Proof. Let $m_+ := \inf\{a; a \in S, a > 0\}$ and $m_- := \inf\{|a|; a \in S, a < 0\}$. We first claim that $m_+ = m_-$. If $a > 0$ and $-b < 0$ are elements of S then $a - nb \in S$ for every $n \in \mathbb{N}$. Hence there is an element $a' \in S$ with $0 < a' \leq b$. So $m_+ \leq m_-$. Similarly $-b + na \in S$ for every $n \in \mathbb{N}$. Hence there is an element $-b' \in S$ with $0 < b' \leq a$. Thus $m_- \leq m_+$.

Now there are two cases. If $m_+ = m_- = 0$ then S is dense in \mathbb{R} . If $m_+ = m_- = a > 0$ we claim that $S = a \cdot \mathbb{Z}$. Suppose $b \in S$. Let $b = na + r$ with $n \in \mathbb{Z}$ and $0 \leq r < a$. Our claim is that $r = 0$. By definition of $m_+ = m_-$ there is an element c of S arbitrarily close to $-na$, hence $b + c \in S$ is arbitrarily close to r , so r must be zero. \square

The case of the additive group \mathbb{R} of theorem 2.1 follows from the following lemma.

Lemma 2.3. *A finitely generated subsemigroup of the additive group \mathbb{R} is either dense in \mathbb{R} or has no accumulation point in \mathbb{R} .*

Proof. Let S be a finitely generated subsemigroup of the additive group \mathbb{R} . If S contains both positive and negative elements, then S is either a discrete cyclic subgroup of \mathbb{R} or S is dense in \mathbb{R} , by lemma 2.2. If S contains only non-negative elements, then there is a minimal element among the positive generators, say s_0 . If t is the number of generators of S then there are at most $t \cdot n$ elements of S of absolute value $\leq n \cdot s_0$. Similarly, if S contains only non-positive elements. \square

Lemma 2.4. *Let S be a finitely generated subsemigroup of \mathbb{R}^n . If \overline{S} has non-empty interior, then the cone $C(S)$ spanned by S , i.e. $C(S) := \{r \cdot s; r \in \mathbb{R}, r \geq 0, s \in S\}$ is dense in \mathbb{R}^n .*

Proof. We first claim that the closure $\overline{C(S)}$ of $C(S)$ is actually a convex cone. Let a, b be elements of S . Then all non-negative integral linear

combinations of a and b are in S hence all non-negative rational linear combinations of a and b are in $C(S)$ and hence all non-negative real linear combinations of a and b are in $\overline{C(S)}$. It follows that the convex cone spanned by two elements of $C(S)$ is contained in $\overline{C(S)}$, therefore $\overline{C(S)}$ is convex.

We now show that $\overline{C(S)}$ is actually all of \mathbb{R}^n . If not there is a linear form l on \mathbb{R}^n such that $l(\overline{C(S)}) \geq 0$, by the separating hyperplane theorem. The subsemigroup $l(S)$ of \mathbb{R} has only non-negative values and, being finitely generated, must be discrete by lemma 2.3. But \overline{S} has non-empty interior hence so does $\overline{l(S)} \supset l(\overline{S})$, since l is an open map. This contradicts our hypothesis. \square

Proof of Theorem 2.1. We first show the theorem for the case $G = \mathbb{R}^n$. Let U be a non-empty open subset of \overline{S} . It suffices to show that \overline{S} intersects $-U$, say $-u \in -U \cap \overline{S}$, $u \in U$, because then the semigroup \overline{S} contains the neighborhood $-u + U$ of zero and hence $\overline{S} = \mathbb{R}^n$. We show in fact that $-U \cap C(S)$ is contained in \overline{S} . Note that the set $-U \cap C(S)$ is non-empty by lemma 2.4. So let $-u \in -U \cap C(S)$. We may assume that $u \neq 0$. Then the closed subsemigroup $\mathbb{R}u \cap \overline{S}$ of the line $\mathbb{R}u$ contains points on either side of zero, hence it is either all of $\mathbb{R}u$ or has no accumulation point, by lemma 2.2. The latter case is impossible since u is an inner point of \overline{S} . Thus $\mathbb{R}u \subset \overline{S}$.

Now let G be a connected abelian Lie group. So there is a covering homomorphism $\pi : \mathbb{R}^n \rightarrow G$ whose kernel is a finitely generated discrete subgroup of \mathbb{R}^n . Hence if S is a finitely generated subsemigroup of G whose closure contains a non-empty open set, then so is $\pi^{-1}(S)$ in \mathbb{R}^n . Hence $\pi^{-1}(S)$ is dense in \mathbb{R}^n , thus S is dense in G .

The general case is now reduced to the case of a connected group by the following lemma and the observation that the image of \overline{S} in G/G^0 , under the quotient map, is a subsemigroup of the finite group G/G^0 and is hence a subgroup. \square

Lemma 2.5. *Let G be an abelian Lie group whose connected component G^0 is of finite index in G . If S is a finitely generated subsemigroup of G which is somewhere dense in G then $S \cap G^0$ contains a finitely generated subsemigroup of G^0 which is somewhere dense in G^0 .*

Proof. Let m be the number of connected components of G . Then $x \mapsto x^m$ is a covering homeomorphism $G \rightarrow G^0$ with finite kernel which maps S to the subsemigroup $S^m = \{s^m; s \in S\}$ of G^0 , which is somewhere dense in G^0 and generated by $\{s_1^m, \dots, s_t^m\}$ if S is generated by $\{s_1, \dots, s_t\}$. \square

We shall need the following fact later on.

Lemma 2.6. *Let G be an abelian Lie group with finite component group G/G^0 .*

- (a) *If G has a dense sub(semi)group with t generators then so does G^0 .*
- (b) *If G^0 has a dense sub(semi)group with t_0 generators and G/G^0 can be generated (as a group or semigroup, there is no difference) by t_1 elements then G can be generated by $\max(t_0, t_1)$ elements.*

Proof. (a) is proved as the preceding lemma.

(b) Let $\bar{g} = gG^0$ be a connected component of G . Suppose it has order $m(\bar{g})$ regarded as an element of the group G/G^0 . Then given an element $h \in G^0$ there is an element $g \in \bar{g}$ such that $g^m = h$ with $m = m(\bar{g})$. This follows from the fact that the group homomorphism $x \mapsto x^m$ of G to itself is a covering with finite kernel and hence maps connected components of G onto connected components of G and that the induced homomorphism $G/G^0 \rightarrow G/G^0$ is $\bar{x} \mapsto \bar{x}^m$. Let us call an element g as above the $m(\bar{g})$ -th root of h and denote it by $\sqrt[m(\bar{g})]{h}$. Our claim now follows from the following fact. Let $\{h_1, \dots, h_{t_0}\}$ be a set of generators of a dense sub(semi)group of G^0 and let $\{\bar{g}_1, \dots, \bar{g}_{t_1}\}$ be a set of generators of G/G^0 . Define $t_2 = \min(t_0, t_1)$ and $A_{\min} = \{\sqrt[m(\bar{g}_1)]{h_1}, \dots, \sqrt[m(\bar{g}_{t_2})]{h_{t_2}}\}$. Then $A_{\min} \cup \{h_{t_2+1}, \dots, h_{t_0}\}$ generates a dense sub(semi)group of G if $t_0 \geq t_1 = t_2$, and $A_{\min} \cup \{g_{t_2+1}, \dots, g_{t_1}\}$ generates a dense sub(semi)group of G if $t_1 \geq t_0 = t_2$, where g_i is an arbitrary element of \bar{g}_i for $i > t_2$. \square

3. PROOF OF THEOREM 1.1.

Observation 3.1. *Let G be an abelian subgroup of $GL(V)$ and let $x \in V$ be a point which has a somewhere dense orbit. Then the isotropy group G_y is trivial for every point y of the orbit $G(x)$ of x .*

Proof. The isotropy group of a point $y = gx$ of the orbit $G(x)$ is $G_y = gG_xg^{-1}$, which equals G_x since G is abelian. So $h \in G_x$ fixes every point of the orbit $G(x)$. But the orbit $G(x)$ spans V , since it is somewhere dense. Hence $h = 1$. \square

All the claims of theorem 1.1, except the dimension of Gx , follow from the following proposition and theorem 2.1.

Proposition 3.2. *Let S be a commutative subsemigroup of $GL(V)$ and let G be its Zariski closure. Suppose that the orbit Sx of $x \in V$ is somewhere dense in V . Then $G \rightarrow Gx$, $g \mapsto gx$, is a diffeomorphism*

of G onto an open subset of V . In particular, S is somewhere dense in G .

In a later paper [1] we will see that Gx is actually dense in V .

Proof of Theorem 1.1. A basic theorem about algebraic actions of algebraic groups says that the orbit $G(x)$ is open in its closure $\overline{G(x)}$ and that the natural map $G \rightarrow G(x)$, $g \mapsto gx$, is a submersion; see [7, § 3.18]. Thus in our case it follows that $G(x)$ contains a non-empty open subset of V . Then $G(x)$ is open in V , by G -invariance. Furthermore, the natural map $G \rightarrow G(x)$ is a diffeomorphism, since it is bijective, by observation 3.1, a submersion by the theorem cited above and its tangent map at the identity element has as kernel the tangent space to G_x which is trivial. It follows that the subsemigroup S of G is somewhere dense in G . \square

4. THE NUMBER OF TOPOLOGICAL GENERATORS OF A CONNECTED ABELIAN LIE GROUP

Let G be a topological group. Let us define $d_{gr}G$ and d_mG to be the minimal number of elements of a subset A of G such that the group, resp. semigroup, generated by A is dense in G . The subscript m was chosen because a semigroup is sometimes called a monoid.

Let G be a connected abelian Lie group. Then G contains a unique maximal compact subgroup T . The subgroup T is actually a torus, i.e. a compact connected abelian Lie group. Let d be the dimension of G/T . The number d is called, occasionally, the *non-compact dimension* of G .

Theorem 4.1.

$$d_{gr}G = d_mG = d + 1$$

unless G is the trivial group.

The proof is broken up into a series of lemmata.

Lemma 4.2 (Kronecker). *For a torus $G \neq \{1\}$ we have $d_{gr}G = d_mG = 1$. More precisely, if $G = \mathbb{R}^n/\mathbb{Z}^n$ then for a vector $v = (r_1, \dots, r_n) \in \mathbb{R}^n$ the monoid generated by $v \bmod \mathbb{Z}^n$ is dense in G if and only if the elements $1, r_1, \dots, r_n$ in \mathbb{R} are linearly independent over \mathbb{Q} .*

Proof. Let H be the closure of the subsemigroup of G generated by $v \bmod \mathbb{Z}^n$. Then H is a subgroup of G since every compact subsemigroup of a topological group is actually a group. Then by Pontryagin duality H is the intersection of the kernels of all continuous homomorphisms $\bar{\varphi} : \mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}/\mathbb{Z}$ which vanish on H . Now $\bar{\varphi}$ is induced

by a linear map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ with the property that $\varphi(\mathbb{Z}^n) \subset \mathbb{Z}$, so $\varphi(x_1, \dots, x_n) = \alpha_1 x_1 + \dots + \alpha_n x_n$ with $\alpha_i \in \mathbb{Z}$ for $(x_1, \dots, x_n) \in \mathbb{R}^n$. But $\varphi = 0$ is the only such map which maps $v = (r_1, \dots, r_n)$ to an element of \mathbb{Z} if and only if $1, r_1, \dots, r_n$ are linearly independent over \mathbb{Q} . \square

Lemma 4.3. $d_{gr}(\mathbb{R}^n) = d_m(\mathbb{R}^n) = n + 1$ for $n > 0$.

Proof. A subgroup of \mathbb{R}^n generated by at most n vectors is clearly not dense in \mathbb{R}^n . On the other hand, we will show that the subsemigroup of \mathbb{R}^n generated by a set A of the following form is dense in \mathbb{R}^n . Let $A = \{v_1, \dots, v_{n+1}\}$, where v_1, \dots, v_n is a basis of the vector space \mathbb{R}^n over \mathbb{R} and $v_{n+1} = r_1 v_1 + \dots + r_n v_n$ and the elements $1, r_1, \dots, r_n$ in \mathbb{R} are linearly independent over \mathbb{Q} and all r_i are negative. To prove this we may assume that v_1, \dots, v_n is the standard basis of \mathbb{R}^n . By lemma 4.2 the subsemigroup of $\mathbb{R}^n/\mathbb{Z}^n$ generated by $v_{n+1} \bmod \mathbb{Z}^n$ is dense in $\mathbb{R}^n/\mathbb{Z}^n$. So for each n -tuple $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and every $\varepsilon > 0$ there is an integer $m \in \mathbb{N}$ and an element $\gamma \in \mathbb{Z}^n$ such that $\|mv_{n+1} + \gamma - x\| < \varepsilon$ for a fixed norm $\|\cdot\|$ on \mathbb{R}^n . We may assume that m is large by applying the preceding lemma to a multiple of v_{n+1} . Then, if $x_i \geq 0$ for $i = 1, \dots, n$, it follows that the coordinates of γ are also non-negative, if ε is sufficiently small. So γ is an element of the subsemigroup \mathbb{N}^n of \mathbb{R}^n generated by v_1, \dots, v_n . Thus every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with $x_i \geq 0$ is contained in the closure H of the subsemigroup of \mathbb{R}^n generated by A . But if $x \in H$ then $x + mv_{n+1} \in H$ for every $m \in \mathbb{N}$. Since all the coordinates of v_{n+1} are negative, by hypothesis, every vector $y \in \mathbb{R}^n$ is of the form $y = mv_{n+1} + x$, for some $m \in \mathbb{N}$ and all coordinates of x non-negative. Hence $H = \mathbb{R}^n$. \square

Proof of Theorem 4.1. Clearly $d_m G \geq d_{gr} G > d$, since G/T is homeomorphic to \mathbb{R}^d . The universal covering group \tilde{G} of G is isomorphic to \mathbb{R}^n for some n and $\pi_1 G$ is isomorphic to a lattice in a subspace of dimension $m = n - d$. Take a basis v_1, \dots, v_n of \mathbb{R}^n containing a basis of $\pi_1 G$ and let $A = \{v_1, \dots, v_n, v_{n+1}\}$ with $v_{n+1} = r_1 v_1 + \dots + r_n v_n$, with $r_i < 0$ and $1, r_1, \dots, r_n$ linearly independent over \mathbb{Q} . Then the subsemigroup of \mathbb{R}^n generated by A is dense in \mathbb{R}^n , hence the subsemigroup of $G = \tilde{G}/\pi_1 G$ generated by the $n + 1 - m = d + 1$ non-zero images of A is dense in G and this finishes the proof. \square

Here is a coordinate free formulation for the conditions on a set A to generate a dense subsemigroup of \mathbb{R}^n which follows from the proof above by choosing appropriate coordinates.

Corollary 4.4. *Let V be a finite dimensional vector space over \mathbb{R} . A finite subset A of V generates a dense subsemigroup of V if and only if the following two conditions hold:*

- (a) *The convex hull of A contains O in its interior.*
- (b) *The zero form is the only \mathbb{R} -linear form l on V with the property that $l(A) \subset \mathbb{Z}$.*

Remark 4.5. In the previous corollary, condition (a) is equivalent to

- (a') *The zero form is the only \mathbb{R} -linear form l on V with the property that $l(A) \subset [0, +\infty)$.*

A more computational version of corollary 4.4 for the case that the cardinality of A is $1 + \dim V$ is the following:

Remark 4.6. A set $\{v_1, \dots, v_{n+1}\}$ of $n+1$ elements of an n -dimensional real vector space V generates a dense subsemigroup of V if and only if the following two conditions hold:

- (a) *The vectors v_1, \dots, v_n are linearly independent over \mathbb{R} .*
- (b) *For the vector $v_{n+1} = \alpha_1 v_1 + \dots + \alpha_n v_n$ we have $\alpha_i < 0$ for all $i = 1, \dots, n$ and $1, \alpha_1, \dots, \alpha_n$ are linearly independent over \mathbb{Q} .*

We will also need the more general case of an arbitrary connected abelian Lie group.

Let G be a connected abelian Lie group and let \mathfrak{g} be its Lie algebra, a real vector space of dimension n , say. The exponential map $\exp : \mathfrak{g} \rightarrow G$ is the universal covering homomorphism of G . Let T be the maximal torus of G and let $\mathfrak{t} = \exp^{-1}(T)$ be its Lie algebra. Then $\Gamma := \ker \exp$ is a lattice in \mathfrak{t} . Let v_1, \dots, v_t be a basis of this lattice.

Corollary 4.7. *Let $\{v_{t+1}, \dots, v_{n+1}\}$ be a set of elements of \mathfrak{g} . Then the subsemigroup of G generated by $\exp(v_{t+1}), \dots, \exp(v_{n+1})$ is dense in G if and only if the following two conditions hold:*

- (a) *The vectors v_1, \dots, v_n form a basis of \mathfrak{g} over \mathbb{R} .*
- (b) *The vector $v_{n+1} = \alpha_1 v_1 + \dots + \alpha_n v_n$ has the following properties: $\alpha_i < 0$ for $i = t+1, \dots, n$ and $1, \alpha_1, \dots, \alpha_n$ are linearly independent over \mathbb{Q} .*

Proof. Let S be the subsemigroup of \mathfrak{g} generated by v_{t+1}, \dots, v_{n+1} . Then $S + \Gamma$ is dense in \mathfrak{g} if the conditions (a) and (b) are satisfied, by remark 4.6, hence $\exp(S)$ is dense in G . Conversely, if S is dense in G , then its image in G/T is dense, hence v_{t+1}, \dots, v_n are linearly independent modulo the span of Γ and $\alpha_i < 0$ for $i > t$. Assume there is a linear relation between $1, \alpha_1, \dots, \alpha_n$ over \mathbb{Q} . So there are numbers b_1, \dots, b_{n+1} in \mathbb{Q} , not all of them zero, such that $b_1 \alpha_1 + \dots + b_n \alpha_n = b_{n+1}$.

We may assume that b_1, \dots, b_{n+1} are in \mathbb{Z} , by multiplying with an appropriate non-zero integer. Now define a linear form $l : \mathfrak{g} \rightarrow \mathbb{R}$ by putting $l(u_i) = b_i$ for $i = 1, \dots, n$. Then $l(u_{n+1}) = b_{n+1}$. So $l(S + \Gamma) \subset \mathbb{Z}$ and hence l induces a non-zero homomorphism $\bar{l} : \mathfrak{g}/\Gamma \rightarrow \mathbb{R}/\mathbb{Z}$ whose kernel contains S . So $\exp S$ is not dense in G , which is isomorphic to \mathfrak{g}/Γ via \exp . \square

5. APPLICATIONS TO FINITELY GENERATED HYPERCYCLIC ABELIAN SEMIGROUPS AND GROUPS OF MATRICES

We first collect the facts we have proven so far in order to apply them to the various cases.

Let V be a real vector space of dimension n and let S be a commutative sub(semi)group of $GL(V)$. Let G be the Zariski closure of S which is a closed abelian subgroup of $GL(V)$ with finite component group and let T be its maximal compact connected subgroup.

Theorem 5.1. *If S has a somewhere dense orbit then S has at least $n + 1 - \dim T$ generators.*

Proof. The group G has dimension n by theorem 1.1, so $d_{gr}G^0 = d_mG^0 = n + 1 - \dim T$ by theorem 4.1 and the corresponding numbers for G are not smaller by lemma 2.6. \square

To obtain the exact minimal number of generators we will apply lemma 2.6, which in the terminology of section 4 reads as follows.

Corollary 5.2. *Let G be an abelian Lie group with finite component group. Then*

$$d_{gr}G = d_mG = \max(d_mG^0, d_mG/G^0) = \max(d_{gr}G^0, d_{gr}G/G^0).$$

Proof. Lemma 2.6 and the formula $d_{gr}G^0 = d_mG^0$ of theorem 4.1. \square

We will apply this for closed abelian subgroups G of $GL(V)$ with finite component group and for which there is a vector $x \in V$ such that the natural map $G \rightarrow G(x)$, $g \mapsto gx$, is a diffeomorphism onto an open subset of V .

Corollary 5.3. *Let G be a closed subgroup of $GL(V)$ with finite component group and let T be a maximal torus of G .*

- (a) *Let S be a commutative sub(semi)group of G with a somewhere dense orbit. Then S has at least $1 + \dim V - \dim T$ generators.*
- (b) *Suppose there is a closed connected abelian subgroup H of G which contains T and has an open orbit. Then there is a dense sub(semi)group S of H with $1 + \dim V - \dim T$ (commuting)*

generators. Every dense subsemigroup S of G has a somewhere dense orbit.

As an application we compute the numbers in tables 1 and 2 of the introduction. *Note that the numbers for groups are the same as for semigroups.* This follows from theorem 4.1. It also turns out that *in all the cases we consider the minimal numbers are the same for a somewhere dense orbit as for a dense orbit.* This is due to corollary 5.2.

Define for any subgroup G of $GL(V)$ the number $m(G)$ to be the minimal number of commuting elements of G with the property that the subsemigroup S of G they generate has a somewhere dense orbit. If G has no finitely generated commutative subsemigroup with a somewhere dense orbit we set $m(G) = \infty$. We have the following results which we list in the form of a table; see table 3. For the definition of T and H in each case see corollary 5.4.

Details will be spelled out in the following corollaries.

For the subsemigroups generated by $m(G)$ elements we have very precise information in case of $G = GL(n, \mathbb{C})$ and $G = GL(n, \mathbb{R})$, see remark 4.6.

TABLE 3.

G	$m(G)$	T	H
$GL(n, \mathbb{C})$	$\mathbf{n} + 1$	$(S^1)^n$	$(\mathbb{C}^*)^n$
$(\mathbb{C}^*)^n \subset G \subset GL(n, \mathbb{C})$	$\mathbf{n} + 1$	$(S^1)^n$	$(\mathbb{C}^*)^n$
$GL(2m, \mathbb{R})$	$\mathbf{m} + 1$	$(SO(2))^m$	$(\mathbb{R}_{>0}^* \cdot SO(2))^m$
$GL(2m + 1, \mathbb{R})$	$\mathbf{m} + 2$	$(SO(2))^m$	$(\mathbb{R}_{>0}^* \cdot SO(2))^m \times \mathbb{R}_{>0}^*$
complex $n \times n$ Toeplitz	$\mathbf{2n}$	S^1	G
real $n \times n$ Toeplitz	$\mathbf{n} + 1$	1	G^0
real triangular	$\mathbf{n} + 1$	1	Toeplitz

Corollary 5.4. *Let G be $GL(n, \mathbb{C})$ or $GL(n, \mathbb{R})$ and let H be the corresponding subgroup of table 3. Let S be a subsemigroup of G generated by $m(G)$ commuting elements and having a somewhere dense orbit. Then the following claims hold:*

- (a) S is diagonalizable over \mathbb{C} .
- (b) S has an everywhere dense orbit unless $G = GL(2m + 1, \mathbb{R})$ in which case for some point the orbit closure is either \mathbb{R}^{2m+1} or a half space of \mathbb{R}^{2m+1} .
- (c) The closure of S in G is a conjugate of H , unless $G = GL(2m + 1, \mathbb{R})$ in which case \overline{S} contains a conjugate H_1 of H and H_1 is of index 1 or 2 in \overline{S} .

We also have the information about the generators formulated in remark 4.6 and corollary 4.7, in case \overline{S} is connected.

Proof. Let us consider the following subgroups of G . For $G = GL(n, \mathbb{C})$ let $H := T_{\mathbb{C}}$ be the group of all diagonal matrices in $GL(n, \mathbb{C})$ and let T be the subgroup of all those matrices in H all of whose entries are of absolute value 1. For $GL(2m, \mathbb{R})$ identify \mathbb{R}^{2m} with \mathbb{C}^m and let T and H be the corresponding subgroups of $GL(m, \mathbb{C}) \subset GL(2m, \mathbb{R})$. For $GL(2m+1, \mathbb{R})$ identify \mathbb{R}^{2m+1} with $\mathbb{C}^m \oplus \mathbb{R}$ and let T be the corresponding subgroup of $GL(m, \mathbb{C}) \times GL(1, \mathbb{R}) \subset GL(2m+1, \mathbb{R})$. For H take $T_{\mathbb{C}} \times \mathbb{R}_{>0}^* \subset GL(m, \mathbb{C}) \times GL(1, \mathbb{R}) \subset GL(2m+1, \mathbb{R})$. Then T and H have the properties of corollary 5.3 and thus prove the first four rows of table 3. In fact, every point of the space has a somewhere dense orbit, except for the points of a finite union of vector subspaces of real codimension 1 or 2. The orbit is actually dense in V except for $G = GL(2m+1, \mathbb{R})$ where its closure is either the whole space or a half space.

Let now S be a subsemigroup of G generated by $m(G)$ commuting elements. Then the Zariski closure G_1 of S has dimension $\dim_{\mathbb{R}} V$ by theorem 1.1 and contains a torus of dimension at least $\dim T$, by theorem 4.1, since $m(G) = 1 + \dim V - \dim T$. It follows that G_1 contains a maximal torus of G . But all maximal tori of G are conjugate. Thus, by conjugating, we may assume that G_1 contains T . Then G_1 is contained in the centralizer of T . But the connected component of the centralizer of T is H in all cases we consider. More specifically, the centralizer of T is H and thus connected, except for $G = GL(2m+1, \mathbb{R})$ where it is $T_{\mathbb{C}} \times \mathbb{R}^*$ and thus contains H of index 2. Now $\dim G_1 = \dim H = \dim V$ and $G_1^0 \subset H$, so $H = G_1^0$. But the closure \overline{S} of S is a subgroup of G_1 and contains G_1^0 , by theorem 1.1, so $\overline{S} = H$ or, for $G = GL(2m+1, \mathbb{R})$, it also may happen that \overline{S} contains H of index 2. This implies all the claims in corollary 5.4. \square

We now deal with the remaining cases of table 3. Let e_1, \dots, e_n be the standard basis of \mathbb{R}^n or \mathbb{C}^n . The *backward shift* σ is the linear map with $\sigma(e_i) = e_{i-1}$ for $i = 2, \dots, n$ and $\sigma(e_1) = 0$. A linear endomorphism is called a Toeplitz operator if it commutes with σ . The following observation is easily checked.

Observation. For a linear endomorphism A of \mathbb{R}^n or \mathbb{C}^n with representing matrix a_{ij} the following properties are equivalent:

- (a) A is Toeplitz.
- (b) A is a linear combination of σ^i , $i = 0, \dots, n-1$, where σ^0 is the identity.

- (c) $a_{i+1,j+1} = a_{ij}$ for all i, j and all $a_{ij} = 0$ for $i > j$.

It follows that the group G of all invertible Toeplitz operators is commutative and has a dense orbit. The group G is connected in the complex case and has two connected components in the real case, corresponding to positive resp. negative diagonal elements. Maximal tori in G are S^1 in the complex case and the trivial group in the real case. This proves the lines with the Toeplitz operators in table 3.

For the group G of triangular matrices, in the complex case G contains the complex diagonal matrices, hence we are in line 2, and in the real case there is no torus. This shows that $m(G) \geq n + 1$ in the real case. The opposite inequality $m(G) \leq n + 1$ follows e.g. from the Toeplitz case.

In a forthcoming paper [1] we will give more precise information on the closure of Sx , if Sx is somewhere dense. It is a union of convex cones of a very special type.

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